COMBINATORICA 1 (3) (1981) 289—301

# THE EXISTENCE OF HOWELL DESIGNS OF SIDE $n+1$ AND ORDER $2n$

by

P. J. SCHELLENBERG, D. R. STINSON
S. A. VANSTONE* and J. W. YATES

University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

*Received 28 October 1980*

A *Howell design* of *side* $s$ and *order* $2n$, or more briefly, an $H(s, 2n)$, is an $s\times s$ array in which each cell either is empty or contains an unordered pair of elements from some $2n$-set, say $X$, such that

(a) each row and each column is Latin (that is, every element of $X$ is in precisely one cell of each row and each column) and

(b) every unordered pair of elements from $X$ is in at most one cell of the array.

A *trivial* Howell design is an $H(s, 0)$ having $X=\emptyset$ and consisting of an $s\times s$ array of empty cells. A necessary condition on $n$ and $s$ for the existence of a nontrivial $H(s, 2n)$ is that $0<n\leqq s\leqq 2n-1$.

An $H(n+t, 2n)$ is said to contain a maximum trivial subdesign if some $t\times t$ subarray is the $H(t, 0)$. This paper describes a recursive construction for Howell designs containing maximum trivial subdesigns and applies it to settle the existence question for $H(n+1, 2n)$'s: for $n+1$ a positive integer, there is an $H(n+1, 2n)$ if and only if $n+1\notin\{2, 3, 5\}$.

## 1. Introduction

A *Howell design* of *side* $s$ and *order* $2n$, or more briefly, an $H(s, 2n)$, is an $s\times s$ array in which each cell either is empty or else contains an unordered pair of elements from a $2n$-set, say $X$, such that

(a) each row and column is Latin (that is, every element of $X$ is in precisely one cell of each row and each column) and

(b) every unordered pair of $X$ is in at most one cell of the array.

A *trivial* Howell design is an $H(s, 0)$ having $X=\emptyset$ and consisting of an $s\times s$ array of empty cells. A necessary condition on $s$ and $n$ for the existence of a nontrivial Howell design is that $0<n\leqq s\leqq 2n-1$.

In the extreme case when $s=2n-1$, an $H(2n-1, 2n)$ is also known as a Room square of side $2n-1$. The existence question for Room squares is completely settled [5]: there is a Room square of side $2n-1$ if and only if $2n-1$ is an odd positive integer, $2n-1\notin\{3, 5\}$.

The existence question for more general Howell designs was first investigated by Hung and Mendelsohn [4]. They showed that in the other extreme case, namely

AMS subject classification (1980): 05 B 30

* St. Jerome's College, University of Waterloo.

when $s=n$, the existence of a pair of orthogonal Latin squares of order $n$ implies the existence of an $H(n, 2n)$. More explicitly, if $A=(a_{ij})$ is a Latin square based on the elements of $\{1, 2, ..., n\}$ and if $B=(b_{ij})$ is an orthogonal Latin square based on the elements of $\{1', 2', ..., n'\}$, then $C=(c_{ij})$, where $c_{ij}=\{a_{ij}, b_{ij}\}$, is an $H(n, 2n)$. Since it is well-known that there are two orthogonal Latin squares of order $n$ for all positive integers $n\neq 2, 6$, it follows that there is an $H(n, 2n)$ for all such $n$. Hung and Mendelsohn also exhibited an $H(6, 12)$. However the main effort of their paper was to establish the existence of an $H(n+k, 2n)$ for $k>1$ provided $n\geqq 2k^2-7k+10$. They showed that for $0\leqq k\leqq 10$, $k\neq 1$, the necessary condition for the existence of a nontrivial $H(n+k, 2n)$ is in fact sufficient with precisely two exceptions, namely an $H(2, 4)$ and an $H(5, 6)$.

For $H(n+1, 2n)$'s, the Howell designs which are investigated in this paper, Hung and Mendelsohn established the following result.

**Lemma 1.1** ([4]).
(i) *For* $n+1$ *even and* $n+1>2$, *there is an* $H(n+1, 2n)$.
(ii) *There does not exist an* $H(3, 4)$ *nor an* $H(5, 8)$.
(iii) *There exists an* $H(7, 12)$ *and an* $H(9, 16)$. ∎

Result (i) was established by exhibiting starters and adders [4] over an Abelian group of order $n+1$. They used a computer search to verify the nonexistence of an $H(5, 8)$. Recently Alex Rosa (private communication) has established this nonexistence result without resorting to the use of a computer. The only other designs in this class which had been shown to exist were an $H(11, 20)$ [6], an $H(13, 24)$ [6] and an $H(17, 32)$ [B. A. Anderson, private communication], all of which were constructed using intransitive starters and adders (see Section 2).

The main result of this paper is that there exists an $H(n+1, 2n)$ iff $n+1$ is a positive integer, $n+1\notin\{2, 3, 5\}$. This is accomplished by describing direct constructions for $H(n+1, 2n)$'s for $n+1\in\{15, 19, 23, 27\}$ (Section 2) and by the use of a recursive construction (Section 3). This recursive construction uses Howell designs which are said to contain the maximum trivial subdesign: an $H(n+t, 2n)$ contains the *maximum trivial subdesign* if some $t\times t$ subarray is the $H(t, 0)$. The new Howell designs produced by this construction also contain the maximum trivial subdesign. In section 4, it is shown that this recursive construction is sufficient to establish the main result of this paper.

An $H(s, 2n)$, based on the elements of a set $X$, is said to satisfy the **-condition* if there exists a subset of $X$, say $Y$, of cardinality $2n-s$ such that no pair of elements from $Y$ is contained in any cell of the design. Most of the direct and recursive constructions which appear in the literature require Howell designs satisfying the *-condition in order to produce larger Howell designs satisfying the *-condition; however, the recursive construction described in this paper does not require Howell designs satisfying the *-condition nor, in general, does it produce Howell designs satisfying the *-condition. Also, the Howell designs constructed from the intransitive starters and adders in Section 2 do not satisfy the *-condition.

Below (see Figure 1.1) we construct an $H^*(13, 24)$. This is the first known *-design in the class $H(n+1, 2n)$, where $n+1$ is odd. (For $n+1$ even, the $H(n+1, 2n)$ of Lemma 1.1 (i) are all *-designs.) We ask if there exist $H^*(n+1, 2n)$ for all odd values of $n+1$ (other than 3 or 5).

The $H^*(13, 24)$ is constructed on the symbol set $(\mathbf{Z}_9 \times \{1, 2\}) \cup \{\alpha, \beta, \gamma, \delta, \varepsilon, \omega\}$. The construction is a modification of the intransitive starter-adder construction described in Section 2. It may be checked that no pair of elements of $(\mathbf{Z}_9 \times \{2\}) \cup \{\alpha, \beta\}$ occurs in the design. Also note the existence of a "subdesign" $H^*(4, 6)$ on the Greek letters.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0_1 0_2$ | $\omega 7_2$ | $5_1 6_1$ | $\varepsilon 5_2$ | $\delta 3_2$ | $\alpha 7_1$ | $\gamma 1_2$ | $\beta 8_1$ | | $1_1 2_2$ | $2_1 4_2$ | $3_1 6_2$ | $4_1 8_2$ |
| | $1_1 1_2$ | $\omega 8_2$ | $6_1 7_1$ | $\varepsilon 6_2$ | $\delta 4_2$ | $\alpha 8_1$ | $\gamma 2_2$ | $\beta 0_1$ | $2_1 3_2$ | $3_1 5_2$ | $4_1 7_2$ | $5_1 0_2$ |
| $\beta 1_1$ | | $2_1 2_2$ | $\omega 0_2$ | $7_1 8_1$ | $\varepsilon 7_2$ | $\delta 5_2$ | $\alpha 0_1$ | $\gamma 3_2$ | $3_1 4_2$ | $4_1 6_2$ | $5_1 8_2$ | $6_1 1_2$ |
| $\gamma 4_2$ | $\beta 2_1$ | | $3_1 3_2$ | $\omega 1_2$ | $8_1 0_1$ | $\varepsilon 8_2$ | $\delta 6_2$ | $\alpha 1_1$ | $4_1 5_2$ | $5_1 7_2$ | $6_1 0_2$ | $7_1 2_2$ |
| $\alpha 2_1$ | $\gamma 5_2$ | $\beta 3_1$ | | $4_1 4_2$ | $\omega 2_2$ | $0_1 1_1$ | $\varepsilon 0_2$ | $\delta 7_2$ | $5_1 6_2$ | $6_1 8_2$ | $7_1 1_2$ | $8_1 3_2$ |
| $\delta 8_2$ | $\alpha 3_1$ | $\gamma 6_2$ | $\beta 4_1$ | | $5_1 5_2$ | $\omega 3_2$ | $1_1 2_1$ | $\varepsilon 1_2$ | $6_1 7_2$ | $7_1 0_2$ | $8_1 2_2$ | $0_1 4_2$ |
| $\varepsilon 2_2$ | $\delta 0_2$ | $\alpha 4_1$ | $\gamma 7_2$ | $\beta 5_1$ | | $6_1 6_2$ | $\omega 4_2$ | $2_1 3_1$ | $7_1 8_2$ | $8_1 1_2$ | $0_1 3_2$ | $1_1 5_2$ |
| $3_1 4_1$ | $\varepsilon 3_2$ | $\delta 1_2$ | $\alpha 5_1$ | $\gamma 8_2$ | $\beta 6_1$ | | $7_1 7_2$ | $\omega 5_2$ | $8_1 0_2$ | $0_1 2_2$ | $1_1 4_2$ | $2_1 6_2$ |
| $\omega 6_2$ | $4_1 5_1$ | $\varepsilon 4_2$ | $\delta 2_2$ | $\alpha 6_1$ | $\gamma 0_2$ | $\beta 7_1$ | | $8_1 8_2$ | $0_1 1_2$ | $1_1 3_2$ | $2_1 5_2$ | $3_1 7_2$ |
| $5_1 1_2$ | $6_1 2_2$ | $7_1 3_2$ | $8_1 4_2$ | $0_1 5_2$ | $1_1 6_2$ | $2_1 7_2$ | $3_1 8_2$ | $4_1 0_2$ | $\alpha\gamma$ | $\varepsilon\delta$ | $\beta\delta$ | |
| $6_1 3_2$ | $7_1 4_2$ | $8_1 5_2$ | $0_1 6_2$ | $1_1 7_2$ | $2_1 8_2$ | $3_1 0_2$ | $4_1 1_2$ | $5_1 2_2$ | | $\alpha\delta$ | $\omega\gamma$ | $\beta\varepsilon$ |
| $7_1 5_2$ | $8_1 6_2$ | $0_1 7_2$ | $1_1 8_2$ | $2_1 0_2$ | $3_1 1_2$ | $4_1 2_2$ | $5_1 3_2$ | $6_1 4_2$ | $\beta\omega$ | | $\alpha\varepsilon$ | $\gamma\delta$ |
| $8_1 7_2$ | $0_1 8_2$ | $1_1 0_2$ | $2_1 1_2$ | $3_1 2_2$ | $4_1 3_2$ | $5_1 4_2$ | $6_1 5_2$ | $7_1 6_2$ | $\delta\varepsilon$ | $\beta\gamma$ | | $\alpha\omega$ |

*Fig. 1.1. An* $H^*(13, 24)$

## 2. Direct constructions

Let $G$ be a finite Abelian group and consider $G \times \{1, 2\}$. For any $(g, i) \in G \times \{1, 2\}$ and for any $h \in G$, define the *sum*

$$(g, i) + h = (g + h, i).$$

For any two elements of $G \times \{1, 2\}$, say $(g, i)$ and $(h, j)$, define the *differences*

$$(g, i) - (h, j) = (g - h, i, j)$$

$$(h, j) - (g, i) = (h - g, j, i).$$

With these definitions in mind, we are in a position to define an intransitive starter and adder. An *intransitive starter* of side $s$ and order $2n$, or more briefly, an $IS(s, 2n)$, over an Abelian group $G$ of order $n$ consists of three lists of pairs from

$G\times\{1,2\}$, namely,

$$R = (\{u_1, v_1\},\ \{u_2, v_2\}, \ldots, \{u_{s-n}, v_{s-n}\})$$
$$C = (\{w_1, x_1\},\ \{w_2, x_2\}, \ldots, \{w_{s-n}, x_{s-n}\})$$
$$S = (\{y_{s-n+1}, z_{s-n+1}\},\ \{y_{s-n+2}, z_{s-n+2}\}, \ldots, \{y_n, z_n\})$$

such that

(a) the $u_i$'s are distinct elements of $G\times\{1\}$ as are the $w_i$'s,
(b) the $v_i$'s are distinct elements of $G\times\{2\}$ as are the $x_i$'s,
(c) the pairs of $C$ and $S$ partition $G\times\{1,2\}$, and
(d) the differences of the pairs in $R$, $C$ and $S$ are all distinct.

An *adder* $A(IS)$ for an $IS(s, 2n)$ is a list of $2n-s$ *distinct* elements of $G$,

$$A(IS) = (a_{s-n+1}, a_{s-n+2}, \ldots, a_n),$$

such that, for

$$T = (\{y_i+a_i, z_i+a_i\} \mid i = s-n+1, s-n+2, \ldots, n),$$

the pairs of $R$ and $T$ partition $G\times\{1,2\}$.

Rosa, Schellenberg and Vanstone have established the following result.

**Lemma 2.1** ([6]). *If there exists an $IS(s, 2n)$ and a corresponding adder $A(IS)$, then there exists an $H(s, 2n)$.* ∎

This result is established by describing the construction of an $H(s, 2n)$ from an $IS(s, 2n)$ and a corresponding adder $A(IS)$. With the aid of a computer, it is possible to construct $IS(n+1, 2n)$'s and corresponding adders $A(IS)$ for small sides $n+1$ and thus establish the existence of the associated Howell designs.

**Lemma 2.2.** *There exists an $H(n+1, 2n)$ for*

$$n+1\in\{7, 9, 11, 15, 17, 19, 23, 27\}.$$

**Proof.** The existence of these Howell designs is established by exhibiting $IS(n+1, 2n)$'s and corresponding adders $A(IS)$ and then applying Lemma 2.1. Each starter $IS(n+1, 2n)$ is constructed over the cyclic group $\mathbf{Z}_n$. For notational convenience, we denote an element $(a, 1)\in\mathbf{Z}_n\times\{1\}$ by $a_1$ and an element $(a, 2)\in\mathbf{Z}_n\times\{2\}$ by $a_2$.

Though Hung and Mendelsohn [4] reported the existence of an $H(7, 12)$ and an $H(9, 16)$, a demonstration of this fact does not appear in the literature. For the sake of completeness, we exhibit an $IS(7, 12)$ and an $IS(9, 16)$ and corresponding adders.

$IS(7, 12)$: $C=(\{6_1, 5_2\})$ $R=(\{4_1, 1_2\})$

| $S$ | $2_1 3_1$ | $1_1 3_2$ | $4_1 4_2$ | $5_1 6_2$ | $1_2 2_2$ |
|---|---|---|---|---|---|
| $A(IS)$ | 3 | 1 | 5 | 2 | 4 |

$IS(9, 16)$: $C=(\{8_1, 2_2\})$ $R=(\{5_1, 1_2\})$

| $S$ | $1_1 4_1$ | $5_1 3_1$ | $6_1 7_1$ | $1_2 4_2$ | $5_2 3_2$ | $6_2 7_2$ | $2_1 8_2$ |
|---|---|---|---|---|---|---|---|
| $A(IS)$ | 8 | 3 | 4 | 2 | 7 | 1 | 5 |

Rosa, Schellenberg and Vanstone [6] exhibited the following intransitive starter.

$IS(11, 20)$: $C=(\{9_1, 6_2\})$ $R=(\{8_1, 3_2\})$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S$ | $10_1 4_1$ | $2_1 5_1$ | $10_2 4_2$ | $7_1 8_1$ | $2_2 5_2$ | $1_1 3_1$ | $7_2 8_2$ | $1_2 3_2$ | $6_1 9_2$ |
| $A(IS)$ | 10 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 |

$IS(15, 28)$: $C=(\{9_1, 4_2\})$ $R=(\{1_1, 2_2\})$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S$ | $1_1 2_1$ | $3_1 5_1$ | $4_1 7_1$ | $6_1 10_1$ | $8_1 13_1$ | $11_1 1_2$ | $12_1 3_2$ | $14_1 2_2$ |
| $A(IS)$ | 1 | 9 | 4 | 13 | 5 | 10 | 12 | 6 |

| | | | | | |
|---|---|---|---|---|---|
| | $5_2 6_2$ | $7_2 10_2$ | $8_2 12_2$ | $9_2 14_2$ | $11_2 13_2$ |
| | 8 | 14 | 11 | 3 | 7 |

B. A. Anderson (private communication) obtained the following $IS(17, 32)$ and $A(IS)$.

$IS(17, 32)$: $C = (\{2_1, 14_2\})$ $R = (\{14_1, 2_2\})$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S$ | $16_1 16_2$ | $6_2 11_2$ | $6_1 12_1$ | $5_1 14_1$ | $1_1 3_1$ | $3_2 2_2$ | $13_2 9_2$ | $10_1 13_1$ |
| $A(IS)$ | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $1_2 8_2$ | $11_1 15_1$ | $12_2 15_2$ | $7_2 5_2$ | $4_1 9_1$ | $10_2 4_2$ | $7_1 8_1$ |
| | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

$IS(19, 36)$: $C = (\{5_1, 2_2\})$ $R = (\{1_1, 2_2\})$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S$ | $1_1 2_1$ | $3_1 6_1$ | $4_1 8_1$ | $7_1 9_1$ | $10_1 15_1$ | $11_1 17_1$ | $12_1 1_2$ | $13_1 3_2$ | $14_1 5_2$ |
| $A(IS)$ | 1 | 3 | 14 | 6 | 2 | 17 | 11 | 12 | 18 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $16_1 4_2$ | $18_1 10_2$ | $6_2 7_2$ | $8_2 11_2$ | $9_2 16_2$ | $12_2 17_2$ | $13_2 15_2$ | $14_2 18_2$ |
| | 13 | 8 | 4 | 5 | 10 | 15 | 9 | 7 |

$IS(23, 44)$: $C = (\{2_1, 13_2\})$ $R = (\{19_1, 6_2\})$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S$ | $1_1 3_1$ | $4_1 5_1$ | $6_1 9_1$ | $7_1 11_1$ | $8_1 13_1$ | $10_1 16_1$ | $12_1 19_1$ | $14_1 22_1$ | $15_1 1_2$ |
| $A(IS)$ | 1 | 2 | 3 | 6 | 12 | 5 | 11 | 18 | 7 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $17_1 2_2$ | $18_1 6_2$ | $20_1 3_2$ | $21_1 5_2$ | $4_2 7_2$ | $8_2 9_2$ | $10_2 14_2$ | $11_2 18_2$ |
| | 19 | 9 | 13 | 17 | 10 | 15 | 21 | 22 |

| | | | | |
|---|---|---|---|---|
| | $12_2 21_2$ | $15_2 20_2$ | $16_2 22_2$ | $17_2 19_2$ |
| | 20 | 14 | 4 | 8 |

$IS(27, 52)$: $C = (\{4_1, 17_2\})$ $R = (\{2_1, 19_2\})$

| $S$ | $1_1 2_1$ | $3_1 5_1$ | $6_1 9_1$ | $7_1 11_1$ | $8_1 13_1$ | $10_1 16_1$ | $12_1 19_1$ | $14_1 22_1$ | $15_1 24_1$ |
|---|---|---|---|---|---|---|---|---|---|
| $A(IS)$ | 2 | 3 | 1 | 4 | 5 | 6 | 7 | 21 | 25 |

| $S$ | $17_1 1_2$ | $18_1 3_2$ | $20_1 2_2$ | $21_1 4_2$ | $23_1 9_2$ | $25_1 5_2$ | $26_1 7_2$ | $6_2 8_2$ | $10_2 11_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $A(IS)$ | 8 | 9 | 26 | 10 | 15 | 22 | 24 | 14 | 19 |

| $S$ | $12_2 15_2$ | $13_2 20_2$ | $14_2 23_2$ | $16_2 22_2$ | $18_2 26_2$ | $19_2 24_2$ | $21_2 25_2$ |
|---|---|---|---|---|---|---|---|
| $A(IS)$ | 11 | 12 | 20 | 17 | 18 | 23 | 16 |

This completes the proof. ∎

## 3. Recursive construction

Let $A=(a_{ij})$ and $B=(b_{ij})$ be a pair of orthogonal Latin squares (OLS) of order $n$ based on the same set of symbols. By the *superposition* of $A$ and $B$ we mean the array $C=(c_{ij})$ where $c_{ij}=(a_{ij}, b_{ij})$. We say that $C$ contains a pair of *orthogonal Latin subsquares* (sub-OLS) of order $m$ if some $m\times m$ subarray of $C$ is itself the superposition of a pair of OLS of order $m$ based on the same symbol set. (The requirement that the sub-OLS be based on the same set of symbols is more restrictive than necessary: however, this rectriction makes some of the definitions that follow much simpler.) Suppose $C$ contains two pairs of sub-OLS, say $D_1$ of order $m_1$ and $D_2$ of order $m_2$. We say that $D_1$ and $D_2$ are *disjoint* sub-OLS (i) if the set of rows and columns which determine $D_1$ is disjoint from the set of rows and columns which determine $D_2$ and (ii) if the set of symbols of $D_1$ is disjoint from that of $D_2$.

As an example of the above terminology, consider the direct product construction of the superposition of a pair of OLS of order 12 from the superposition of a pair of OLS of order 3 and a pair of order 4. Let

$$U = \begin{vmatrix} 11 & 22 & 33 \\ 23 & 31 & 12 \\ 32 & 13 & 21 \end{vmatrix} \quad \text{and} \quad V = \begin{vmatrix} 11 & 34 & 42 & 23 \\ 43 & 22 & 14 & 31 \\ 24 & 41 & 33 & 12 \\ 32 & 13 & 21 & 44 \end{vmatrix}$$

For $i, j \in \{1, 2, 3, 4\}$, define $U_{ij}$ to be the array obtained from $U$ by replacing each pair $(a, b)$ of $U$ by the pair $(a_i, b_j)$. Then

$$W = \begin{array}{|c|c|c|c|} \hline U_{11} & U_{34} & U_{42} & U_{23} \\ \hline U_{43} & U_{22} & U_{14} & U_{31} \\ \hline U_{24} & U_{41} & U_{33} & U_{12} \\ \hline U_{32} & U_{13} & U_{21} & U_{44} \\ \hline \end{array}$$

is the superposition of a pair of OLS of order 12. According to the above definitions, $U_{11}$ is a pair of sub-OLS of $W$ whereas $U_{34}$ is not: furthermore, $U_{11}$ and $U_{22}$ are disjoint sub-OLS.

A *spanning set* of disjoint sub-OLS of $C$ is a set $\{D_1, D_2, ..., D_r\}$ of disjoint sub-OLS of orders $m_1, m_2, ..., m_r$, respectively, such that $m_1+m_2+...+m_r=n$. This implies that every row (column) of $C$ passes through precisely one of the $D_i$'s and every symbol of $C$ is in precisely one of the $D_i$'s. In the above example, $\{U_{11}, U_{22}, U_{33}, U_{44}\}$ is a spanning set of disjoint sub-OLS. If this spanning set is removed from $W$ to obtain

$W' =$

| | | | |
|---|---|---|---|
| $\varphi$ | $U_{34}$ | $U_{42}$ | $U_{23}$ |
| $U_{43}$ | $\varphi$ | $U_{14}$ | $U_{31}$ |
| $U_{24}$ | $U_{41}$ | $\varphi$ | $U_{12}$ |
| $U_{32}$ | $U_{13}$ | $U_{21}$ | $\varphi$ |

where $\varphi$ represents a 3 by 3 array of empty cells, we say that $W'$ is *missing* a spanning set of disjoint sub-OLS. The superposition of a pair of OLS missing a spanning set of disjoint sub-OLS is an example of what we now define as a partitioned pair of incomplete orthogonal Latin squares.

Let $X$ be any set and let $\mathcal{G}=\{G_1, G_2, ..., G_r\}$ be a partition of $X$ into subsets $G_1, G_2, ..., G_r$ called *parts*. By a *partitioned pair of incomplete orthogonal Latin squares* (*ILS*), based on the set $X$ and the partition $\mathcal{G}$, or more briefly, an $ILS(X, \mathcal{G})$, we mean an $|X|$ by $|X|$ array whose rows and columns are labelled with the elements of $X$ such that

(i) for $1 \leqq i \leqq r$, the subarray determined by the rows and columns labelled with the elements of $G_i$ has every cell empty;

(ii) every other cell contains a single pair from $(X \times X) \setminus \left( \bigcup_{i=1}^{r} (G_i \times G_i) \right)$ and each of these pairs is in exactly one cell of the array; and

(iii) for any $x \in G_i \subset X$, row $x$ and column $x$ are both Latin in $X \setminus G_i$ (that is, the pairs in row $x$ constitute a bijection from $X \setminus G_i$ onto itself as do the pairs in column $x$).

We are now in a position to describe the main recursive construction of this paper. Recall that an $H(n+t, 2n)$ contains the maximal trivial subdesign if some $t \times t$ subarray is the $H(t, 0)$.

**Theorem 3.1.** *Let $t$ be a positive integer. If, for some set $X$ and some partition $\mathcal{G}=\{G_1, G_2, ..., G_r\}$, there is an $ILS(X, \mathcal{G})$ and if, for each part $G_i$, there is an $H(|G_i|+t, 2|G_i|)$ containing the maximum trivial subdesign $H(t, 0)$, then there is an $H(|X|+t, 2|X|)$ containing the maximum trivial subdesign.*

**Construction.** Let $A$ be an $ILS(X, \mathcal{G})$. Without loss of generality, we may assume the first $|G_1|$ rows and columns are labelled with the elements of $G_1$, the next $|G_2|$ with the elements of $G_2$ and so on until the final $|G_r|$ rows and columns are labelled with the elements of $G_r$. Let $B$ be the array obtained from $A$ by replacing each pair

$(a, b)$ of $A$ by the unordered pair $\{a_1, b_2\}$. Then array $B$ can be partitioned as follows:

$$B = \begin{array}{|c|c|c|c|} \hline \varphi_1 & B_{12} & \cdots & B_{1r} \\ \hline B_{21} & \varphi_2 & \cdots & B_{2r} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline B_{r1} & B_{r2} & \cdots & \varphi_r \\ \hline \end{array}$$

where $\varphi_i$ represents a $|G_i|$ by $|G_i|$ subarray of empty cells and the $B_{ij}$'s contain all the pairs $\{a_1, b_2\}$ where $(a, b) \in (X \times X) \setminus \left( \bigcup_{i=1}^{r} (G_i \times G_i) \right)$.

By hypothesis, for each part $G_i$ there is an $H(|G_i|+t, 2|G_i|)$ containing the maximum trivial subdesign. Without loss of generality, we may assume that

$$H_i = \begin{array}{|c|c|} \hline \varphi & J_i \\ \hline K_i & L_i \\ \hline \end{array}$$

is such a Howell design based on the elements of $\{a_1, a_2 | a \in G_i\}$ where $\varphi$ represents the maximum trivial subdesign $H(t, 0)$.

From $B$ we obtain the array $B'$ by replacing $\varphi_i$ by $L_i$. By adjoining the $t \times |X|$ subarray

$$[J_1 | J_2 | \ldots | J_r]$$

to the top of $B'$ we get $B''$. By adjoining the $(|X|+t) \times t$ subarray

$$\left[ \begin{array}{c} \varphi \\ \hline K_1 \\ \hline K_2 \\ \hline \vdots \\ \hline K_r \end{array} \right]$$

to the left of $B''$ we obtain

$$C = \begin{array}{|c|c|c|c|c|} \hline \varphi & J_1 & J_2 & \cdots & J_r \\ \hline K_1 & L_1 & B_{12} & \cdots & B_{1r} \\ \hline K_2 & B_{21} & L_2 & \cdots & B_{2r} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline K_r & B_{r1} & B_{r2} & \cdots & L_r \\ \hline \end{array}$$

which is an $H(|X|+t, 2|X|)$ containing the maximum trivial subdesign $\varphi$.

**Proof.** Consider any row, say row $x$, from the subarray

| | | | | | | |
|---|---|---|---|---|---|---|
| $K_i$ | $B_{i1}$ | $B_{i2}$ | $\dots$ | $L_i$ | $\dots$ | $B_{ir}$ |

where $1 \leqq i \leqq r$. Since the corresponding row in the $ILS(X, \mathcal{G})$ is Latin in $X \setminus G_i$, row $x$ contains all the elements of $\{a_1, a_2 | a \in X \setminus G_i\}$ exactly once in the cells of the $B_{ij}$, $j = 1, 2, \dots, r$, $j \neq i$. Since

$$H_i = \begin{array}{|c|c|} \hline \varphi & J_i \\ \hline K_i & L_i \\ \hline \end{array}$$

is an $H(|G_i| + t, 2|G_i|)$ row $x$ contains all the elements of $\{a_1, a_2 | a \in G_i\}$ exactly once in cells of $K_i$ and $L_i$. Hence row $x$ is Latin as required.

Next consider any row, say row $h$, from the subarray

| | | | | |
|---|---|---|---|---|
| $\varphi$ | $J_1$ | $J_2$ | $\dots$ | $J_r$ |

The portion of row $h$ in $J_i$, $1 \leqq i \leqq r$, is Latin in the symbols of $\{a_1, a_2 | a \in G_i\}$ since $H_i$ is a Howell design. Hence row $h$ is Latin as required.

Similarly, any column of $C$ is Latin.

From the definition of an $ILS(X, \mathcal{G})$ and from the construction of array $B$ from $A$ it follows that $B$ contains every pair of the form $\{a_1, b_2\}$, where $(a, b) \in (X \times X) \Big/ \left( \bigcup_{i=1}^{r} (G_i \times G_i) \right)$, precisely once. Let $P$ be the set of all these pairs. For $1 \leqq i \leqq r$, each unordered pair of distinct elements from the set $\{a_1, a_2 | a \in G_i\}$ is in at most one cell of $H_i$. None of these pairs is in $P$ and, furthermore, for $1 \leqq j \leqq r$, $j \neq i$, the pairs in array $H_i$ are distinct from all of those in $H_j$. Thus no unordered pair of elements from $\{a_1, a_2 | a \in X\}$ is contained in more than one cell of $C$.

This establishes that $C$ is an $H(|X| + t, 2|X|)$ containing the maximum trivial subdesign. ∎

In order to apply this Theorem, we need to establish the existence of *ILS*. The following two lemmata describe constructions for such arrays.

**Lemma 3.2.** *If there exists a pair of OLS based on the elements of $N$ having a common transversal and if there exists a pair of OLS based on the elements of $M$, then there exists an $ILS(X, \mathcal{G})$ where $X = M \times N$ and $\mathcal{G} = \{M \times \{a\} | a \in N\}$.*

**Proof.** Recall that a common transversal of the superposition of a pair of OLS based on the set $N$ is a collection of cells, one from each row and each column, such that the pairs in these cells constitute a bijection of $N$ onto itself. By permuting rows and/or columns and/or symbols it is always possible to have the common transversal on the main diagonal and to have the pairs of the form $(a, a)$ where $a \in N$.

By the direct product construction [3] for a pair of OLS, it is possible to construct a pair of OLS of order $mn$ having a spanning set of disjoint sub-OLS of order $m$. For example, the pair of OLS of order 12 above were constructed in this fashion. Observe that the common transversal in the pair of OLS of order $n$ guarantees the existence of the spanning set of disjoint sub-OLS of order $m$. Removing this spanning set of disjoint sub-OLS produces an $ILS(X, \mathcal{G})$ where $X = M \times N$ and $\mathcal{G} = \{M \times \{a\} | a \in N\}$. ▮

The next lemma requires the notion of a group divisible design. A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$ where (i) $X$ is a set whose elements are called *points*, (ii) $\mathcal{G}$ is a class of subsets of $X$ (called *groups*) which partition $X$, (iii) $\mathcal{A}$ is a class of subsets of $X$ (called *blocks*), each of cardinality at least two, (iv) each group and each block intersect in at most one point and (v) each pair of distinct points which is not contained in any group is contained in precisely one block of $\mathcal{A}$.

If $\mathcal{G}$ has $s_i$ groups of cardinality $m_i$, where $i = 1, 2, \ldots, p$, then $(m_1^{s_1}, m_2^{s_2}, \ldots, m_p^{s_p})$ is the *group type* of $(X, \mathcal{G}, \mathcal{A})$.

**Lemma 3.3.** *If there is a $GDD(X, \mathcal{G}, \mathcal{A})$ and if, for each block $A \in \mathcal{A}$, there exists a pair of OLS of order $|A|$ having a common transversal then there is an $ILS(X, \mathcal{G})$.*

**Proof.** This is essentially the construction for sets of pairwise OLS described by Bose, Shrikhande and Parker [Theorem 1, 1].

For any block $A \in \mathcal{A}$, let $[A]$ denote a pair of OLS, based on the set $A$, having a common transversal: furthermore the rows and columns of $[A]$ are labelled with the elements of $A$. By permuting rows and/or columns and/or symbols, we may assume that the common transversal consists of the pairs $(a, a)$, for all $a \in A$, which are contained in the cells $(a, a)$ respectively. From $[A]$, obtain the array $[A]'$ by deleting all the pairs of the common transversal.

Let $Y$ denote an $|X|$ by $|X|$ array whose rows and columns are labelled with the elements of $X$. For any block $A \in \mathcal{A}$ let $A \times A$ denote that subarray of $Y$ determined by the rows and columns which are labelled with the elements of $A$. Now, for each block $A \in \mathcal{A}$, place the array $[A]'$, described above, in the subarray $A \times A$ of $Y$. The result can be shown to be an $ILS(X, \mathcal{G})$. ▮

We are now in a position to establish the existence of Howell designs $H(n+1, 2n)$ for all positive integers $n+1 \notin \{2, 3, 5\}$.

## 4. The Existence of $H(n+1, 2n)$'s

We begin this section by listing some results from the literature on combinatorial designs which are required in the proofs of Lemmata 4.4, 4.5 and 4.6 below.

**Lemma 4.1** ([1]). *There is a pair of OLS of positive integral order $n$ if and only if $n \notin \{2, 6\}$.* ▮

**Lemma 4.2** ([2]). *There is a pair of OLS of positive integral order $n$ having a common transversal if and only if $n \notin \{2, 3, 6\}$.* ▮

A *transversal design* with $k$ groups of cardinality $n$, or more briefly, a $TD(k, n)$ is a $GDD(X, \mathcal{G}, \mathcal{A})$ which has $k$ groups, each of cardinality $n$, and which has each block of cardinality $k$. This immediately implies there are $n^2$ blocks and each point of $X$ is in precisely $n$ blocks of $\mathscr{A}$. It is well-known (see for example [3]) that the existence of a $TD(k, n)$ is equivalent to the existence of $(k-2)$ pairwise OLS of order $n$.

**Lemma 4.3.** ([8]). *If $m$ is a positive integer such that $m \notin \{2, 3, 6, 10, 14\}$, then there is a $TD(5, m)$.* ∎

**Lemma 4.4.** *If $n=4q+r$ where $r \leqq q$, if there is $TD(5, q)$ and if there is an $H(q+1, 2q)$ and an $H(r+1, 2r)$, then there is an $H(n+1, 2n)$.*

**Proof.** Remove $q-r$ points from one group of a $TD(5, q)$ to obtain a $GDD(X, \mathscr{G}, \mathscr{A})$ with group type $(q^4 r^1)$ and having block sizes 4 and 5. By Lemmata 3.3 and 4.2 there is an $ILS(X, \mathscr{G})$. By hypothesis there is an $H(q+1, 2q)$ and an $H(r+1, 2r)$ both of which contain the maximum trivial subdesign and hence Theorem 3.1 implies the existence of an $H(n+1, 2n)$. ∎

**Lemma 4.5.** *If $1 \leqq n+1 \leqq 65$ and $n+1 \notin \{2, 3, 5\}$, then there is an $H(n+1, 2n)$. If $n+1 \in \{2, 3, 5\}$ there does not exist an $H(n+1, 2n)$.*

**Proof.** If $n+1=1$, an $H(n+1, 2n)$ is the trivial $H(1, 0)$. If $n+1=2$, an $H(n+1, 2n)$ does not exist since the necessary conditions on the side and the order are not satisfied. If $n+1 \in \{3, 5\}$, there does not exist an $H(n+1, 2n)$ by Lemma 1.1. If $n+1$ is even $4 \leqq n+1 \leqq 64$, then there is an $H(n+1, 2n)$ by Lemma 1.1. By Lemma 2.2, there is an $H(n+1, 2n)$ for $n+1 \in \{7, 9, 11, 15, 17, 19, 23, 27\}$.

To complete this Lemma we make repeated use of Theorem 3.1. Observe that every $H(n+1, 2n)$ contains the maximum trivial subdesign $H(1, 0)$.

Let $n+1=4m+1 \geqq 13$ where $m$ is odd. By Lemmata 3.2, 4.1 and 4.2, there is an $ILS(X, \mathscr{G})$ where $|X|=4m$ and each part of $\mathscr{G}$ has cardinality $m$. Since $m>1$ is odd, there is an $H(m+1, 2m)$ containing a maximum trivial subdesign by Lemma 1.1 and hence Theorem 3.1 implies there is an $H(4m+1, 8m)$.

The following table describes how to establish the existence of some additional $ILS(X, \mathscr{G})$'s.

| $\lvert X\rvert$ | Authority | Parameters | |
|---|---|---|---|
| 24 | Lemma 3.2 | $\lvert N\rvert = 8$, | $\lvert M\rvert = 3$ |
| 30 | Lemma 3.2 | $\lvert N\rvert = 10$, | $\lvert M\rvert = 3$ |
| 32 | Lemma 3.2 | $\lvert N\rvert = 4$, | $\lvert M\rvert = 8$ |
| 40 | Lemma 3.2 | $\lvert N\rvert = 8$, | $\lvert M\rvert = 5$ |
| 42 | Lemma 3.2 | $\lvert N\rvert = 14$, | $\lvert M\rvert = 3$ |
| 48 | Lemma 3.2 | $\lvert N\rvert = 16$, | $\lvert M\rvert = 3$ |
| 50 | Lemma 3.2 | $\lvert N\rvert = 10$, | $\lvert M\rvert = 5$ |
| 54 | Lemma 3.2 | $\lvert N\rvert = 18$, | $\lvert M\rvert = 3$ |
| 56 | Lemma 3.2 | $\lvert N\rvert = 8$, | $\lvert M\rvert = 7$ |
| 64 | Lemma 3.2 | $\lvert N\rvert = 8$, | $\lvert M\rvert = 8$ |

In each case Theorem 3.1 together with Lemma 1.1 implies the existence of the corresponding $H(n+1, 2n)$, where $n=|X|$.

There remain five cases to consider; namely $n+1 \in \{35, 39, 47, 59, 63\}$.

Applying Lemma 4.4, where $q$ and $r$ are as defined in the accompanying table,

| $n+1$ | $q$ | $r$ |
|---|---|---|
| 35 | 7 | 6 |
| 39 | 8 | 6 |
| 59 | 13 | 6 |
| 63 | 13 | 10 |

together with Lemmata 1.1, 2.2 and 4.3, it follows that there is an $H(n+1, 2n)$ for $n+1 \in \{35, 39, 59, 63\}$.

Finally, to establish the existence of an $H(47, 92)$, we construct an $ILS(X, \mathcal{G})$ where $|X|=46$ and $\mathcal{G}$ has one part of cardinality 10 and four of cardinality 9. This is accomplished by the use of the direct singular product [7]. Let $C$ represent the superposition of a pair of OLS of order 10 based on the elements of $\{0, 1, 2, \ldots, 9\}$. Without loss of generality, we may assume the pair $(0, 0)$ is in the upper left-hand corner square and that $C$ is partitioned as follows:

$C =$

| | |
|---|---|
| (0, 0) | $T$ |
| $L$ | $R$ |

where $T$ is a $1 \times 9$ submatrix, $L$ is $9 \times 1$ and $R$ is $9 \times 9$. Let $N$ be the superposition of a pair of OLS of order 9 based on the elements of $\{1, 2, \ldots, 9\}$. For $A$ equal to any of $T$, $L$ or $N$, define $A_{ij}$ to be the array obtained from $A$ by replacing each pair $(a, b)$ of $A$ by the pair $(a_i, b_j)$. $R_{ij}$ is defined somewhat differently: for $a \neq 0 \neq b$, replace pair $(a, b)$ by pair $(a_i, b_j)$, replace $(a, 0)$ by $(a_i, 0)$ and $(0, b)$ by $(0, b_j)$. Then

$D =$

| | | | | | |
|---|---|---|---|---|---|
| (0, 0) | $T_{11}$ | $T_{54}$ | $T_{42}$ | $T_{35}$ | $T_{23}$ |
| $L_{11}$ | $R_{11}$ | $N_{45}$ | $N_{24}$ | $N_{53}$ | $N_{32}$ |
| $L_{35}$ | $N_{43}$ | $N_{22}$ | $N_{51}$ | $R_{35}$ | $N_{14}$ |
| $L_{54}$ | $N_{25}$ | $R_{54}$ | $N_{33}$ | $N_{12}$ | $N_{41}$ |
| $L_{23}$ | $N_{52}$ | $N_{31}$ | $N_{15}$ | $N_{44}$ | $R_{23}$ |
| $L_{42}$ | $N_{34}$ | $N_{13}$ | $R_{42}$ | $N_{21}$ | $N_{55}$ |

is a pair of OLS of order 46 containing a spanning set of disjoint sub-OLS; namely,

| | |
|---|---|
| (0, 0) | $T_{11}$ |
| $L_{11}$ | $R_{11}$ |

$N_{22}$, $N_{33}$, $N_{44}$ and $N_{55}$. By removing this spanning set of disjoint sub-OLS we obtain an $ILS(X, \mathcal{G})$ where

$$X = \{0\} \cup \{a_i | a \in \{1, 2, ..., 9\},\ i \in \{1, 2, 3, 4, 5\}\},$$

$$G_1 = \{0\} \cup \{a_1 | a \in \{1, 2, ..., 9\}\}$$

and, for $2 \leqq i \leqq 5$,

$$G_i = \{a_i | a \in \{1, 2, ..., 9\}\}.$$

Then by Lemmata 1.1 and 2.2 and by Theorem 3.1, there is an $H(47, 92)$. ∎

**Lemma 4.6.** *If $n+1>65$, then there is an $H(n+1, 2n)$.*

**Proof.** For any integer $n$, there exist unique integers $q$ and $r$ such that $n=4q+r$ where $5 \leqq r \leqq 8$. For $5 \leqq r \leqq 8$, we establish the existence of an $H(4q+r+1, 8q+2r)$ by mathematical induction over $q$. Lemma 4.4 implies the existence of such Howell designs for $1 \leqq q \leqq 14$.

For $5 \leqq r \leqq 8$, assume there is an $H(4q+r+1, 8q+2r)$ for $1 \leqq q \leqq Q-1$ where $Q \geqq 15$. Now consider $n=4Q+r$, $5 \leqq r \leqq 8$. By the induction hypothesis there is an $H(Q+1, 2Q)$ and by Lemma 1.1 there is an $H(r+1, 2r)$. Since $Q \geqq 15$, Lemma 4.3 implies there is a $TD(5, Q)$. Hence Lemma 4.4 implies there is an $H(n+1, 2n)$.

Thus, by mathematical induction, we have that, for $5 \leqq r \leqq 8$, there is an $H(4q+r+1, 8q+2r)$ for all positive integers $q$. ∎

Combining the last two Lemmata, we get

**Theorem 4.7.** *For $n+1$ a positive integer, there is an $H(n+1, 2n)$ if and only if $n+1 \notin \{2, 3, 5\}$.*

**Addendum.** Recently, Anderson et al. [9] and Stinson [10] have completely settled the existence question for all other types of Howell designs.

## References

[1] R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Can. J. Math.* **12** (1960), 189—203.

[2] R. K. Brayton, D. Coppersmith and A. J. Hoffman, Self-orthogonal Latin squares of all orders $n \neq 2, 3, 6$. *Bull. AMS* **80** (1974), 116—118.

[3] M. Hall, Jr., *Combinatorial Theory*, Blaisdell, Waltham, Mass. (1967).

[4] S. H. Y. Hung and N. S. Mendelsohn, On Howell designs, *J. Combinatorial Theory (A)* **16** (1974), 174—198.

[5] R. C. Mullin and W. D. Wallis, The existence of Room squares *Aequationes Math.* **13** (1975), 1—7.

[6] A. Rosa, P. J. Schellenberg and S. A. Vanstone, Generalized starter and adder constructions for Howell designs, *Proc. 10th Southeastern Conf. on Combinatorics, Graph Theory and Computing (vol. II)*, Boca Raton, Fla. (1979), 833—842.

[7] A. Sade, Produit direct-singulier de quasigroupes orthogonaux et anti abéliens, *Ann. Soc. Sci. Bruxelles, Sér. I*, **74** (1960), 91—99.

[8] S. M. P. Wang and M. Wilson, A few more squares, II (abstract), *Proc. 9th Southeastern Conf. on Combinatorics, Graph Theory and Computing*, Boca Raton, Fla. (1978), 688.

[9] B. A. Anderson, P. J. Schellenberg and D. R. Stinson, The existence of Howell designs of even side, *preprint*.

[10] D. R. Stinson, The existence of Howell designs of all side, J. Combinatorial Theory, *to appear*.